Exercise 0

Show that the set of all dilation-translations $v \mapsto \lambda v + b$ with $\lambda \neq 0$ of an affine plane form a group under composition.

Let A be set of all dilation transformations

$$v \mapsto \lambda v + b \quad \lambda \neq 0$$

claim: A is a group with operation composition.

Let $D_1, D_2 \in A$

such that $D_1(v) = \lambda v + b, \lambda \neq 0$

$D_2(v) = \lambda' v + b', \lambda' \neq 0$

so; $(D_2 \circ D_1)v = D_2(D_1(v)) = D_2(\lambda v + b)$

$= \lambda'(\lambda v + b) + b'$

$= \lambda\lambda' v + \lambda' b + b'$

$(D_2 \circ D_1)(v) = (\lambda\lambda')v + (\lambda' b + b')$

$\lambda \neq 0$ and $\lambda' \neq 0$

so $\lambda\lambda' \neq 0$

Exercise 0

Hence $D_2 \circ D_1 \in A$

So A is close under composition

Now set $D_e \in A$ such that

$$D_e(v) = v$$

So for any $D_i \in A$

$$(D_e \circ D_i)(v) = D_i(D_e(v)) = D_i(v)$$

Hence

$$D_e \circ D_i = D_i \circ D_e = D_i \quad \forall D_i \in A$$

So D_e is the identity element.

Set $D \in A$ be any

$$D(v) = \lambda v + b, \quad \lambda \neq 0$$

Let us define

$$D^{-1}(v) = \frac{1}{\lambda} v - \frac{b}{\lambda}$$

So clearly $D^{-1} \in A$

$$(D^{-1} \circ D)(v) = D^{-1}(D(v)) = D^{-1}(\lambda v + b)$$

$$= \frac{1}{\lambda}(\lambda v + b) - \frac{b}{\lambda}$$

$$= v = D_e(v)$$

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Exercise o

$$(D \circ D^{-1})(v) = D(D^{-1}(v)) = D\left(\frac{1}{\lambda}v - \frac{b}{\lambda}\right)$$

$$= v = D_e(v).$$

Hence

$$D \circ D^{-1} = D^{-1} \circ D = D_e.$$

so D^{-1} is the inverse of D

so each element in A has inverse.

Now

Let $D_1(v) = \lambda_1 v + b_1$

$D_2(v) = \lambda_2 v + b_2$

$D_3(v) = \lambda_3 v + b_3$

so

$$D_1 \circ (D_2 \circ D_3) v = D_1(D_2 \circ D_3(v))$$

$$= D_1(D_2(D_3(v)))$$

$$= D_1(\lambda_2\lambda_3 v + \lambda_3 b_3 + b_2)$$

$$= \lambda_1\lambda_2\lambda_3 v + \lambda_1\lambda_2 b_3 + \lambda b_2 + b_1$$

and $(D_1 \circ D_2) \circ D_3(v) = (D_1 \circ D_2)(D_3(v))$

$$= D_1(D_2(\lambda_3 v + b_3))$$

$$= \lambda_1\lambda_2\lambda_3 v + \lambda_1\lambda_2 b_3 + \lambda_1 b_2 + b_1$$

$$= D_1 \circ (D_2 \circ D_3) v$$

Exercise 0

Hence

$D_1 \circ (D_2 \circ D_3) = (D_1 \circ D_2) \circ D_3$

so, associativity Holds in A

Hence A is a group.

Exercise 1

Let $D_1(v) = 2(v - \begin{bmatrix}1\\1\end{bmatrix}) + \begin{bmatrix}1\\1\end{bmatrix}$ be dilation in $\mathbb{R}^2$

Let $D_2(v) = \lambda(v-p) + p$

So $D_1(v) = 2(v - \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}$

$= 2v - \begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}$

$= 2v - \begin{bmatrix}1\\1\end{bmatrix}$

Now $D_2 \circ D_1(v) = D_2(D_1(v)) = D_2(2v - \begin{bmatrix}1\\1\end{bmatrix})$

$= \lambda(2v - \begin{bmatrix}1\\1\end{bmatrix} - p) + p$

Let $p = \begin{pmatrix}x\\y\end{pmatrix}$

So $D_2 \circ D_1(v) = \lambda(2v - \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}x\\y\end{bmatrix}) + \begin{bmatrix}x\\y\end{bmatrix}$

$= 2\lambda v - \lambda \begin{bmatrix}1+x\\1+y\end{bmatrix} + \begin{bmatrix}x\\y\end{bmatrix}$

$= 2\lambda v + \begin{bmatrix}x - \lambda - \lambda x\\ y - \lambda - \lambda x\end{bmatrix}$

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Exercise 1

but $D_2 \circ D_1(v) = v + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So,

$$2\lambda v + \begin{bmatrix} x - \lambda - \lambda x \\ y - \lambda - \lambda x \end{bmatrix} = v + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we get

$2\lambda = 1 \longrightarrow \lambda = \frac{1}{2}$

and $x - \lambda - \lambda x = 1$

$x - \frac{1}{2} - \frac{1}{2}x = 1$

$x = 3$

$y - \lambda - \lambda y = 0$

$y - \frac{1}{2} - \frac{1}{2}y = 0$

$y = \underline{1}$

Hence $P = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$D_2(v) = \frac{1}{2}(v - \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$=$

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Exercise 2

F^2

$\mathbb{F} = \{(a,b) : a, b \in F_3\}$ with $(a,b) +$
$(a'b') = (a+a', b+b')$ and $(a,b)(a'b') =$
$(aa' - bb'd, ab' + a'b)$ of elements

How many point are in $\mathbb{F}^2$?

F_3^2 is a cartesian product $q = 3 \cdot 3$ points
Line is given by $q + \lambda v$ where
$v \in F_3$. There are many points
on each lines there are element
in F_3 i.e 3 points
Pick a point different from origin.
There is one line through the
origin and it's chosen point
So we have 8 lines $(q-1) = 8$
There are 4 lines through the
origin.
Therefore Total lines. (9×8)
$= 72$ lines.

Exercise 2

Each line is unique $3! = 6$ times
There are 3 points on each line
Hence $72/6 = 12$ lines.

(1 How many points are in F_3^2?
~~72 points.~~
$\cancel{72}/2$ points $= 36$ points

(2 How many lines in F_3^2?
12 lines

(3 How many lines are in F_3^2 that pass through the origin $(0,0)$?
4 lines

(4 How many points lie on each line

3 points on each line

Q3 Exercise 3

vertices $p_1 = (1, 2), p_2 = (2, 3)$
and $p_3 (4, 1)$
F_5^2 over the field F_5
with 5 elements
$P_i P_j$ divided by t_{ij}
$T_{12} = 1, T_{23} = 2$ and $T_{31} = 3$

$P_1 t_{23}, P_2 t_{31}$ and $P_3 t_{12}$ concurrent?
If yes determine their intersection?

Solution

Assume that the vertices of the triangle have the coordinates
$P_1 = (1, 0, 0)$ $P_2 = (0, 1, 0)$, $P_3 (0, 0, 1)$
in affine F^3. Then t_{ij} have coordinates

$$T_{12} = \frac{1}{1+T_{12}} (1, T_{12}, 0)$$

$$T_{23} = \frac{1}{1+T_{23}} (0, 1, T_{23})$$

$$T_{31} = \frac{1}{1+T_{31}} (T_3, 0, 1)$$

Exercise 3

Lines through t_{ij} and P_k are given as the intersection of the triangle plane $x_1 + x_2 + x_3 = 1$ with the planes

$$T_{12}\, x_1 = x_2$$
$$T_{23}\, x_2 = x_3$$
$$T_{31}\, x_3 = x_1$$
$$T_{12} = 1 \quad T_{23} = 2 \quad T_{31} = 3$$

$$T_{12} = \frac{1}{1+1}(1,1,0) \Rightarrow \frac{1}{2}(1,1,0)$$

$$T_{23} = \frac{1}{1+2}(0,1,2) \Rightarrow \frac{1}{3}(0,1,2)$$

$$T_{31} = \frac{1}{1+3}(3,1,0) \Rightarrow \frac{1}{4}(3,1,0)$$

Assume the 3 lines are concurrent. Therefore there is common intersection point (x_1, x_2, x_3) of the 4 planes: and its coordinates satisfy all the 4 equation

Exercise 3

$x_i \neq 0$

$x_1 + x_2 + x_3 = 1$ multiply both sides

$T_{12} T_{22} T_{31} = 1$. you get.

$T_{12} T_{23} T_{31} = 1$ it is not parallel because at least two points intersect in point q.

Assume the lines $p_1 t_{23}$ then q satisfy the equation $T_{23} x_2 = x_3$ and $p_2 t_{31} \to T_{31} x_3 = x_1$

$p_1 \to (1, 2)$ $T_{23} = 2$

$(1, 2)2 \to 2x_2 = x_3$

$T_{12} T_{23} T_{31} = 1$ it satisfy the first equation $T_{12} X_1 = X_2$ and q lies on $p_3 t_{12}$.

Therefore bijective affine map from affine plane $x_1 + x_2 + x_3 = 1$ to F^2.